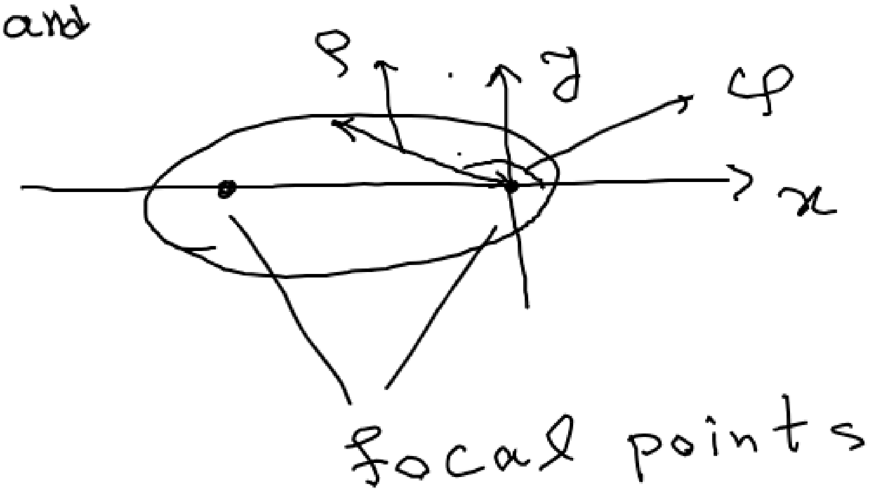
OK, in the last session, we finally arrived at an equation for the elleipse in polar coordinates.

The axes were chose so that one of the focal points of the ellipse is on the origin, the major axis of the ellipse is on the x axis~~. is on~~ and the other focal point is on the negative part of the x axis:



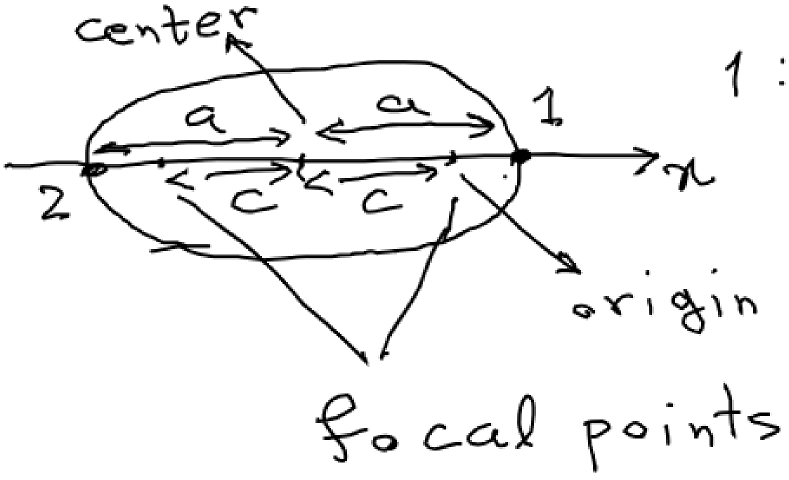
$$\rho = \frac{b^2}{a(1+e\cos\varphi)}$$

$$e = \frac{c}{a}$$

↓ eccentricity

$$b^2 = a^2 - c^2 = a^2(1-e^2)$$

$$\rho = \frac{a(1-e^2)}{1+e\cos\varphi}$$



1: $\varphi = 0$

$$\rho = \frac{a(1-e^2)}{1+e}$$

$$= a(1-e)$$

$$= a-c$$

$$\rho = \frac{a(1-e^2)}{1+e\cos\varphi}$$

2: $\varphi = \pi$

$$\rho = \frac{a(1-e^2)}{1-e}$$

$$= a(1+e)$$

$$= a+c$$

1: The nearest point to the origin. For the sun at the origin, this point is called the perihelion.

2: The farthest point to the origin. For the sun at the origin, this point is called the aphelion.

If the origin is oocupied by the earth (for example the elleipse is the orbit of the moon around the earth, the points are called perigee and apogee, respectively.

farthest

nearest

According to Kepler's first law, the orbit of a planet around the sun is an elleipse, the equation of which was obtained (in terms of the polar coordinates).

What does this empirical law tell about the force between the sun and the planet?

Newton's second law states that the equation of motion for the planet is this.

$$\vec{F} = m\vec{a}$$

$m \rightarrow$ mass of the planet

$\vec{a} \rightarrow$ acceleration of the planet

$\vec{F} \downarrow$ the force applied by the sun on the planet

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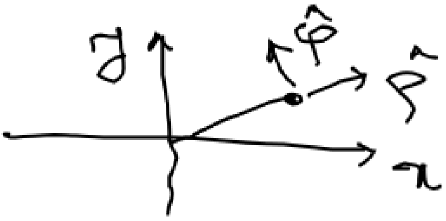
In terms of the polar coorniates:

$$\vec{F} = F_\rho \hat{\rho} + F_\varphi \hat{\varphi}$$

$$\vec{a} = a_\rho \hat{\rho} + a_\varphi \hat{\varphi}$$



Two scalar equations

$F_\rho = m a_\rho \qquad F_\varphi = m a_\varphi$

One has to express the radial and azimuthal components od the acceleration in terms of the coordinates and their derivatives.

The position vector is $\vec{r}$ in terms of the Cartesian coordinate

$\vec{r} = x\hat{x} + y\hat{y} \longleftarrow$

The acceleratin is the second derivative of the position:

$\ddot{\vec{r}} = \ddot{x}\hat{x} + \ddot{y}\hat{y}$, as $\hat{x}$ and $\hat{y}$ are constant vectors.

Expressing the Cartesian coordinates in terms of the polar coordinates,

$x = \rho\cos\varphi, \quad y = \rho\sin\varphi$ Differentiating with respect to the time,

$$\dot{x} = \dot{\rho}\cos\varphi - (\rho\sin\varphi)\dot{\varphi}$$

$$\dot{y} = \dot{\rho}\sin\varphi + (\rho\cos\varphi)\dot{\varphi}$$

$$\ddot{x} = \ddot{\rho}\cos\varphi - 2\dot{\rho}\dot{\varphi}\sin\varphi - (\rho\cos\varphi)\dot{\varphi}^2 - (\rho\sin\varphi)\ddot{\varphi}$$

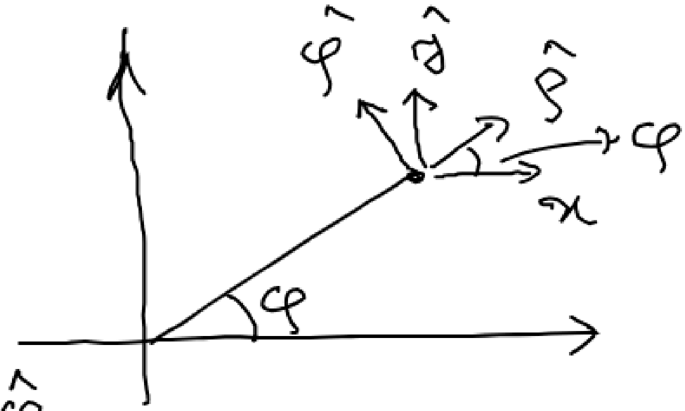
$$\ddot{y} = \ddot{\rho}\sin\varphi + 2\dot{\rho}\dot{\varphi}\cos\varphi - (\rho\sin\varphi)\dot{\varphi}^2 + (\rho\cos\varphi)\ddot{\varphi}$$

$$\vec{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y}$$

$$= (\ddot{\rho} - \rho\dot{\varphi}^2)(\hat{x}\cos\varphi + \hat{y}\sin\varphi)$$

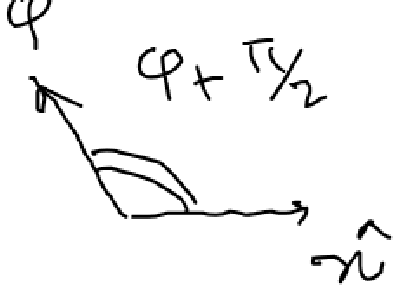
$$+ (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})(-\hat{x}\sin\varphi + \hat{y}\cos\varphi)$$

This the acceleration in terms of the polar coordinates and their derivatives, but Cartesian unit vectors.



$$\hat{\rho} = \hat{x}\cos\varphi + \hat{y}\sin\varphi$$

$$\hat{\varphi} = \hat{x}\cos(\varphi + \pi/2) + \hat{y}\sin(\varphi + \pi/2)$$



$$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$$

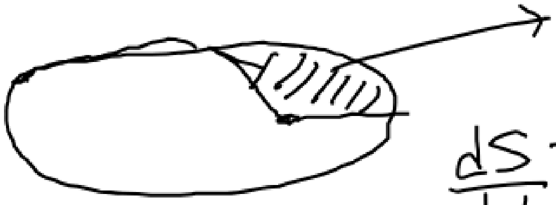
$$\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{\varphi}$$
$$= a_\rho\hat{\rho} + a_\varphi\hat{\varphi}$$

$$F_\rho = m(\ddot{\rho} - \rho\dot{\varphi}^2) = m a_\rho$$

$$F_\varphi = m(\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi}) = m a_\varphi$$

These equations contain information about the shape of the orbit,
and also on how the planet moves on the orbit: the time dependences of rho and phi.
In order to obtain an equation for the shape (only), one has to eliminate the time,
so that a relation between rho and phi is found.

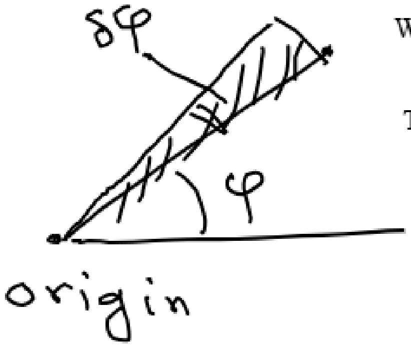
Before that (eliminating the time), let's recall the second law of Kepler.



This surface area is a function of time that its derivative
is a constant.

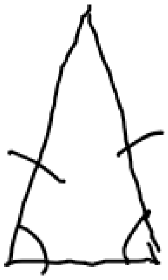
$$\frac{dS}{dt} = \text{Constant}$$

Let's express this surface area in terms of the polar coordinates.



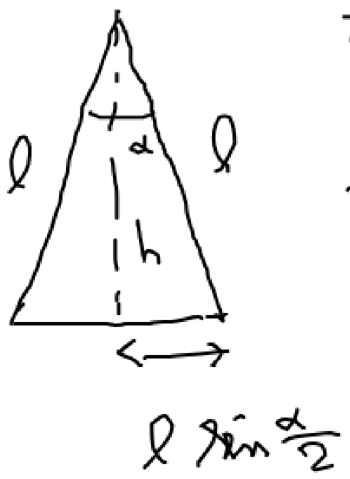
What is the surface area of the shaded region?

The shaded region is approximately an isosceles triangle.



isosceles triangle: two side lengths are the sam

(Which means that the corresponding angles are the same



the base length $= 2l\sin\frac{\alpha}{2}$

the height $= h = l\cos\frac{\alpha}{2}$

the surface area $= S$

$= \frac{1}{2}$(the base length) $\times$ height

$= l^2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \frac{l^2}{2}\sin\alpha$

This is the exact expression

What happens if alpha is small?

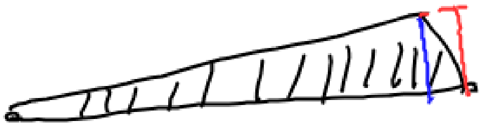
if α is small, $\sin\alpha \approx \alpha$

more rigorously: $\lim_{\alpha\to 0} \frac{\sin\alpha}{\alpha} = 1$

$S \approx \frac{\ell^2}{2}\alpha$

more rigorously: $\lim_{\alpha\to 0} \frac{S(\alpha)}{\frac{\ell^2\alpha}{2}} = 1$

The surface the area of which is to be calculated, is a region bounded between the orbit and two rays:



This area is between two areas: the blue and the red ones.

The blue area corresponds to an isosceles triangle with $l = r_{min}$

The red area correspods to an isasceles triangle with $h = r_{max}$

$S_{blue} \leq S \leq S_{red}$ $\qquad l = \frac{r_{max}}{\cos\frac{\alpha}{2}}$

$$\frac{S_{blue}}{\frac{r_{min}^2}{2}\alpha} \longrightarrow 1 \qquad \frac{S_{red}}{\frac{r_{max}^2 \alpha}{2\cos^2\frac{\alpha}{2}}} \longrightarrow 1 \qquad \frac{S}{\frac{\bar{r}^2\alpha}{2}}$$

$\bar{r}$ is between r_{min} and r_{max}

$$\frac{S_{blue}}{\frac{\bar{r}^2\alpha}{2}} \le \frac{S}{\frac{\bar{r}^2\alpha}{2}} \le \frac{S_{red}}{\frac{\bar{r}^2\alpha}{2}}$$

$$\lim_{\alpha\to 0}$$

$$\lim_{\alpha\to 0} \frac{S_{blue}}{\frac{\bar{r}^2\alpha}{2}} = \lim_{\alpha\to 0} \overset{\to 1}{\boxed{\frac{S_{blue}}{\frac{r_{min}^2\alpha}{2}}}} \overset{\to 1}{\boxed{\frac{r_{min}^2}{\bar{r}^2}}}$$

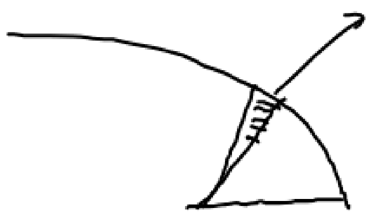
$\alpha\to 0$ r_{min} and r_{max} → the same and so does $\bar{r}$ as $\bar{r}$ is between r_{min} and r_{max}

So: $\lim_{\alpha \to 0} \frac{S_{blue}}{\frac{\bar{r}^2 \alpha}{2}} = 1$

$$\frac{S_{red}}{\frac{\bar{r}^2 \alpha}{2}} = \underbrace{\frac{S_{red}}{\frac{r_{max}^2 \alpha}{2 Cs^2 \frac{\alpha}{2}}}}_{\to 1} \times \overbrace{\frac{r_{max}^2}{\bar{r}^2}}^{\to 1} \times \overbrace{\frac{1}{Cs^2 \frac{\alpha}{2}}}^{\to 1}$$

$$\lim_{\alpha \to 0} \frac{S_{red}}{\frac{\bar{r}^2 \alpha}{2}} = 1 \qquad \frac{S_{blue}}{\frac{\bar{r}^2 \alpha}{2}} \leq \frac{S}{\frac{\bar{r}^2 \alpha}{2}} \leq \frac{S_{red}}{\frac{\bar{r}^2 \alpha}{2}}$$

$$\Rightarrow \lim_{\alpha \to 0} \frac{S}{\frac{\bar{r}^2 \alpha}{2}} = 1$$



$$\frac{S}{\frac{\bar{r}^2\alpha}{2}} \to 1$$

α is the angle corresponding to the shaded are

What is the relation between $\bar{r}$ and r (or $\bar{\rho}$ and ρ)

$$\lim_{\alpha\to 0} \frac{\bar{\rho}}{\rho} = 1 \quad \text{So: } \lim_{\alpha\to 0} \frac{S}{\frac{\rho^2\alpha}{2}} = 1$$

$$S \longrightarrow \Delta S \qquad \lim_{d\varphi \to 0} \frac{\Delta S}{\frac{\rho^2 \Delta\varphi}{2}} = 1$$

$$\alpha \longrightarrow \Delta\varphi$$

$$\lim_{\Delta\varphi \to 0} \frac{\Delta S}{\Delta\varphi} = \frac{\rho^2}{2} \qquad \frac{dS}{d\varphi} = \frac{\rho^2}{2}$$

dS

Kepler's second law: [(dS) / (dt)] is a constant. It doesn't say that [(ds) /(d phi)] is a constant.

One can use the chain rule to express [(ds) / (dt)] in terms of [(dS) / (d phi)]:

$$\frac{dS}{dt} = \frac{dS}{d\varphi}\frac{d\varphi}{dt} = \frac{\rho^2}{2}\frac{d\varphi}{dt} \qquad \dot{S} = \frac{\rho^2 \dot{\varphi}}{2}$$

Kepler's second law: $\frac{\rho^2\dot{\varphi}}{2}$ = constant

Newton's second law:

$\dot{S}$ = constant

$\ddot{S} = 0$

$$F_\rho = m(\ddot{\rho} - \rho\dot{\varphi}^2)$$

$$F_\varphi = m(\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})$$

$$\frac{d}{dt}\left(\frac{\rho^2\dot{\varphi}}{2}\right) = \rho\dot{\rho}\dot{\varphi} + \frac{\rho^2\ddot{\varphi}}{2}$$

$$\frac{\rho^2\dot{\varphi}}{2} = \frac{dS}{dt} = \dot{S}$$

$$F_\varphi = m\,\frac{2}{\rho}\,\frac{d}{dt}\left(\frac{\rho^2\dot{\varphi}}{2}\right)$$

$$F_\varphi = \frac{2m}{\rho}\,\frac{d}{dt}\dot{S} = \frac{2m}{\rho}\ddot{S}$$

The φ component of Newton's second law: $F_\varphi = \frac{2m}{\rho} \ddot{S}$

Kepler's second law: $\ddot{S} = 0$

The two combined $\Rightarrow$ $F_\varphi = 0$

For this result ($F_\varphi = 0$) Kepler's first (or third) law is not needed.

Newton's second law + Kepler's second law

$$\vec{F} = F_\rho \hat{\rho} \quad \Longleftarrow$$

The force is a radial force

F_ρ (the radial component of the force) could in principle depend on both ρ and φ, but for gravity, it doesn't depend on φ.

If an orbit is just rotated (in the same plane around the sun) another acceptable orbit is obtained.

This can be seen, for examplefrom Kepler's third law.

This symmetry means that if the position is rotated, the force is rotated by the same rotation as well.
This is why the radial component of the force doesn't depend on phi.
So finally,

$$\vec{F}(\vec{r}) = [F_\rho(\rho)]\,\hat{\rho}$$

This is what so far was obtained for the gravitational force