1 A very simple model for movement

Someone starts moving, say running or swimming. The motion begins from rest. That is, at $t = 0$ (T is the time), the speed v is 0. But the mover has an acceleration, so as time passes the speed increases. However, that cannot continue indefinitely. The mover gets tired and the speed begins to decrease. It could happen that eventually the mover stops.

Let's try building a very simplified model. The distance covered by the mover (from the beginning of motion, at the time zero) is denoted by x. The aim is to build a time dependence for x, such that x increases with time but at the infinite-time limit tends to a constant value. It is also required that the speed v (the first derivative of x) be zero at $t = 0$, but positive for $t > 0$, and the acceleration be positive at $t = 0$.

Taylor-expanding x in t around $t = 0$, one has

$$x(t) = x(0) + [\dot{x}(0)]\, t + [\ddot{x}(0)]\, \frac{t^2}{2} + o(t^2), \tag{1}$$

where $o(t^2)$ means something that tends to zero faster that t^2 (as t tends to zero). $x(0)$ is zero, as the distance is calculated from $t = 0$. $\dot{x}(0)$ is also zero, as the speed is zero at $t = 0$. So,

$$x(t) = \frac{a\, t^2}{2} + o(t^2), \tag{2}$$

where a is the acceleration at $t = 0$. Note that the acceleration is not constant. One can rewrite the above as

$$x(t) = \frac{a\, t^2}{2}\, f(t). \tag{3}$$

This is in fact the definition of f: $f(t)$ has been defined as the ratio of $x(t)$ to $(a\, t^2/2)$. To complete the model, one has to determine f.

Of course f is not unique. But some things are known about it. It is known that f tends to 1 as t tends to 0, so that as small times $x(t)$ is essentially $(a\, t^2/2)$. It is also known that at t to infinity, $f(t)$ behaves like t^{-2}, so that $x(t)$ tends to a constant as t tends to infinity, as demanded. A very simple form of f satisfying these is

$$f(t) = \frac{1}{1 + b\, t^2}, \tag{4}$$

where b is a positive constant. One then arrives at the following simplified model for x.

$$x(t) = \frac{a\, t^2}{2\, (1 + b\, t^2)}. \tag{5}$$

This models contains just two parameters a and b. a has the dimension of acceleration (in fact it is the acceleration at $t = 0$), b has the dimension of one over time squared. So one can write it in the form τ^{-2}, where τ has the dimension of time. In terms of this, and ℓ with

$$\ell = \frac{a\,\tau^2}{2}, \tag{6}$$

The expression for x becomes

$$x(t) = \ell\,\frac{(t/\tau)^2}{1 + (t/\tau)^2}. \tag{7}$$

Obviously, ℓ in the limit of $x(t)$ as t tends to infinity. It is also seen that $x(t)$ is increasing with t. So this very simple model has all of the properties which were required.

It is easy to calculate v (the instantaneous speed) and $\bar{v}$ (the average speed from the time zero): $v(t)$ is the time derivative of x at t, and $\bar{v}(t)$ is the ratio of $x(t)$ to t:

$$v(t) = \frac{\ell}{\tau}\,\frac{2\,(t/\tau)}{[1 + (t/\tau)^2]^2}. \tag{8}$$

$$\bar{v}(t) = \frac{\ell}{\tau}\,\frac{(t/\tau)}{1 + (t/\tau)^2}. \tag{9}$$

There is a time t_{s} at which $v(t)$ is maximized, and a time t_{m} at which $\bar{v}(t)$ is maximized. These are the time at which the time derivatives of v and $\bar{v}$, respectively, vanishes. Differentiating v and $\bar{v}$ and setting the derivatives equal to zero, one arrives at

$$1 - 3\left(\frac{t_{\mathrm{s}}}{\tau}\right)^2 = 0. \tag{10}$$

$$1 - \left(\frac{t_{\mathrm{m}}}{\tau}\right)^2 = 0. \tag{11}$$

So,

$$t_{\mathrm{s}} = \frac{\tau}{\sqrt{3}}. \tag{12}$$

$$t_{\mathrm{m}} = \tau. \tag{13}$$

To determine the parameters τ and ℓ, some data is needed. Suppose that the average speeds at t_1 and t_2 are equal to the same value V. Find ℓ and τ.

As a numerical example, consider the case of running competitions for short distances. The (approximate) world records for the 100 m and 200 m races are 10 s and 20 s, respectively. What are τ and ℓ? What are the maximum values of the instantaneous and average speeds?