1 How to measure the number of molecules, III

By now, two apparently unrelated phenomena has been discussed under the title of measuring the number of molecules, and yet no method have been presented to actually measure the number of molecules.

Let's return to scattering. Consider a beam of light passing through the atmosphere (or whatever scattering medium). The intensity of light is its power per surface area normal to the path of the light. Consider a light moving in the direction x, and consider a thin slab of material between x and $(x + \Delta x)$, having the surface area A. The intensity of the light entering the slab is $I(x)$. So the power entering the slab is $[A\, I(x)]$. Of this power, some is scattered. The scattered power is equal to the number of scatterers times the power scattered from one scatterer (assuming the medium is dilute, so that the scatterers don't affect each other). The power which leaves the slab, is equal to the power which had entered the slab, minus the scattered power. So, using the expression which had been found for the power scattered by one scatterer, one arrives at

$$A\, I(x + \Delta\, x) = A\, I(x) - \frac{b\, N\, K\, c\, p^2}{\lambda^4}, \tag{1}$$

where b is a dimesnionless constant and N is the number of molecules in the slab. The volume of the slab is equal to $(A\, \Delta\, x)$. Bringing the first term on the right-hand side to the left, dividing the result by the volume V, and sending $(\Delta\, x)$ to zero, one arrives at

$$\frac{\mathrm{d}\, I}{\mathrm{d}\, x} = -\frac{b\, N\, K\, c\, p^2}{V\, \lambda^4}. \tag{2}$$

Recall that the electric dipole of a molecule is proportional to the electric field:

$$p = \varepsilon_0\, \gamma\, E. \tag{3}$$

So,

$$\frac{\mathrm{d}\, I}{\mathrm{d}\, x} = -\frac{b\, N\, K\, \varepsilon_0^2\, c\, \gamma^2\, E^2}{V\, \lambda^4}. \tag{4}$$

It can be shown that the intensity of light is related to its electric field as follows.

$$I = \varepsilon_0\, c\, E^2. \tag{5}$$

Putting this in (4), one arrives at

$$\frac{\mathrm{d}\, I}{\mathrm{d}\, x} = -b\, (K\, \varepsilon_0)\, \frac{N\, \gamma^2}{V}\, \frac{I}{\lambda^4}. \tag{6}$$

$(K\,\varepsilon_0)$ is equal to $(4\,\pi)^{-1}$. Absorbing this in b, the above relation is rewritten as

$$\frac{\mathrm{d}\,I}{\mathrm{d}\,x} = -b'\,\frac{N\,\gamma^2}{V}\,\frac{I}{\lambda^4}. \tag{7}$$

The coefficient of I on the right-hand side, has the dimension of length inverse. Denoting this by L^{-1}, the equation is simplified to

$$\frac{\mathrm{d}\,I}{\mathrm{d}\,x} = -\frac{I}{L}. \tag{8}$$

The solution to this, is

$$I(x) = I(0)\,\exp\left(-\frac{x}{L}\right). \tag{9}$$

This means that the intensity of light (in fact this could be done for any wave), is attenuated as the light moves in the medium, and the attenuation is exponential. The parameter L is called the attenuation length. It is the length that when light travels a distance equal to that, its intensity is multiplied by e^{-1}. So L could be determined, using macroscopic measurements. One has

$$\frac{1}{L} = \frac{b'}{\lambda^4}\,\frac{N\,\gamma^2}{V}. \tag{10}$$

Now recall the expression for the susceptibility:

$$\chi = \frac{N\,\gamma}{V}. \tag{11}$$

Measuring L and χ, one could determine both $(N\,\gamma^2)$ and $(N\,\gamma)$, and hence both N and γ. This way one can actually count the number of molecules in a sample, through macroscopic measurements. Continuous matter (no discrete particles) corresponds to the limit N to infinity, while $(N\,\gamma)$ is kept fixed. It is seen that in this limit γ tends to zero, hence L tends to infinity, meaning that there is no light scattering. If the matter was continuous (not made of discrete particles) the sky of the earth would have been black.