1 Scaling

Consider two quantities X and Y, assuming that Y depends on X:

$$Y = f(X). \tag{1}$$

What happens to Y if the value of X is changed from x to $(\alpha\, x)$? Of course the new value of Y would be $f(\alpha\, x)$, and one would know little about $f(\alpha\, x)$ or its relation with $f(x)$, the former value of Y, unless something is known about f. In fact if x_0 and y_0 are two constants with dimensions equal to those of X and Y, respectively, one could write the dependence of Y on X as

$$Y = y_0\, \tilde{f}(X/x_0), \tag{2}$$

and $\tilde{f}$ could be any function with a dimensionless variable resulting in a dimensionless value.

However, for this to happen, two constants with proper dimensions are needed. A constant (x_0) is needed with which X is compared, and another (y_0) is needed with which Y is compared. So a small or large value for X or Y is meaningful, and these are compared with x_0 and y_0, respectively.

It is seen that if x_i corresponds to y_i, then

$$\frac{y_2}{y_1} = \frac{\tilde{f}(x_2/x_0)}{\tilde{f}(x_1/x_0)}. \tag{3}$$

The right-hand side depends on x_0, in general. So the relative change of Y depends on x_0 as well.

What happens if there is no constant x_0, so that it is meaningless to say the value of X is small or big? The one expects that if the value of X is multiplied by α, no matter what the initial value of X has been, the effect of the value of Y be *the same*. But what is meant by *the same*?

To answer these questions, let's demand that the right-hand side of (3) be independent of x_0. So it should be a function of only x_1 and x_2:

$$\frac{\tilde{f}(x_2/x_0)}{\tilde{f}(x_1/x_0)} = g(x_1, x_2). \tag{4}$$

But as the left-hand side is independent of x_0, one could use any value for x_0 in it. In particular, one could use x_1 instead of x_0. Then,

$$\frac{\tilde{f}(x_2/x_1)}{\tilde{f}(1)} = g(x_1, x_2). \tag{5}$$

This shows that $g(x_1, x_2)$ is a function of only the ratio (x_2/x_1):

$$g(x_1, x_2) = h(x_2/x_1). \tag{6}$$

So,

$$\tilde{f}(u) = \tilde{f}(1)\, h(u), \tag{7}$$

and, using (4),

$$h(x_2/x_0) = h(x_1/x_0)\, h(x_2/x_1), \tag{8}$$

which can be rewritten as

$$h(u\, v) = h(u)\, h(v). \tag{9}$$

The solution to this equation, assuming that h is continuous, is

$$h(u) = u^a, \tag{10}$$

where a is a constant. The final result for the dependence of Y on X, would then be

$$Y = c\, X^a, \tag{11}$$

where

$$c = \frac{y_0\, \tilde{f}(1)}{x_0^a}. \tag{12}$$

So, if there are no constants to compare X and Y with which, then the dependence of Y on X should be a power law: Y is proportional to X to the power of some constant. Of course this is the case if such a dependence does exist. Such behavior is called scaling. As some examples,

- The dependence of the surface area on the length is a power law. What is the exponent?
- The dependence of the volume on the length is a power law. What is the exponent?